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Solutions of nonlinear PDEs in the sense of averages

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Abstract

We characterize p-harmonic functions including $p = 1$ and $p = \infty$ by using mean value properties extending classical results of Privaloff from the linear case $p = 2$ to all p's. We describe a class of random tug-of-war games whose value functions approach p-harmonic functions as the step goes to zero for the full range $1 < p < \infty$.

Résumé

On caractérise les fonctions p-harmoniques, y compris les cas $p = 1$ at $p = \infty$, en utilisant des propriétés de la moyenne. Ces résultats prolongent le cas classique linéaire ($p = 2$) dû à Privaloff, à toutes les valeurs de p. Pour tout p dans l'intervalle $(1, \infty)$, on décrit une classe de jeux aléatoires de type « tous à la corde » dont les fonctions valeur approchent les fonctions p-harmoniques lorsque le pas tend vers zero.

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1. Introduction

In this article we study solutions to a class of nonlinear equations that can be characterized by mean value properties. The quintessential example is the characterization of harmonic functions by the property

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\,dy. \tag{1}$$

Privaloff [24] proved that an upper-semicontinuous function u is subharmonic if and only if

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$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\Bigg[\fint_{B_\varepsilon(x)} u(y)\,dy - u(x)\Bigg] \geqslant 0. \tag{2}$$

A similar statement for spherical means was obtained by Blaschke [1].

If we replace the Laplace equation $\Delta u = 0$ by a linear elliptic equation with constant coefficients $Lu = \sum_{i,j} a_{ij} u_{x_i x_j} = 0$ then mean value formulas now hold for appropriate ellipsoids instead of balls. This is true also in the subelliptic case. See Chapter 5 in the recent book [3] and the paper [2] for updated discussion of mean value properties for solutions of linear equations.

We are interested in understanding mean value properties in the nonlinear case. We start by observing that in order to characterize continuous harmonic functions it is enough to ask that the mean value property (1) holds in an asymptotic sense

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\,dy + o\big(\varepsilon^2\big) \quad \text{as } \varepsilon \to 0. \tag{3}$$

In fact, even a weaker *viscosity* notion suffices. An upper-semicontinuous function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *subharmonic* in Ω if for every $x \in \Omega$ and test function $\phi \in C^2(\Omega)$ that touches u from above at x we have that

$$\phi(x) \leqslant \fint_{B_\varepsilon(x)} \phi(y)\,dy + o\big(\varepsilon^2\big) \quad \text{as } \varepsilon \to 0. \tag{4}$$

Notice that the characterization (4) implies a simple proof of one half of Privaloff's characterization. Similarly, solutions to the p-Laplace equation are characterized by

$$u(x) = \frac{p-2}{2(p+n)}\Big\{\max_{\bar B_\varepsilon(x)} u + \min_{\bar B_\varepsilon(x)} u\Big\} + \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u(y)\,dy + o\big(\varepsilon^2\big) \tag{5}$$

in the viscosity sense, for p in the range $1 < p \leqslant \infty$. These facts are proven in [19]. That is, we have the analogue of Privaloff's characterization for p-subharmonic functions by replacing the regular solid average with the nonlinear average in (5) and using expansions in the viscosity sense. For a related evolution problem see [7] as well as [20], and for general discussion of p-Laplacian problems including $p = 1$ and $p = \infty$ see [13].

In [22], Peres and Sheffield showed that p-harmonic functions are limits of value functions of certain tug-of-war games with noise as the step size tends to zero. These games were modified in [21] so that their value function u_ε, with step size $\varepsilon > 0$, was uniquely defined, and satisfied a dynamic programming principle of the form

$$u_\varepsilon(x) = \frac{p-2}{2(p+n)}\Big\{\max_{\bar B_\varepsilon(x)} u_\varepsilon + \min_{\bar B_\varepsilon(x)} u_\varepsilon\Big\} + \frac{n+2}{p+n} \fint_{B_\varepsilon(x)} u_\varepsilon(y)\,dy,$$

when $p \geqslant 2$. See also [15,16] and [23].

The objectives of this paper are to consider the limit case $p = 1$, to characterize the 1-harmonic functions in the spirit of (5), and to obtain a dynamic programming principle valid for all $p > 1$ for the corresponding tug-of-war game. Theorem 9 below states that 1-harmonic subsolutions are characterized by

$$u(x) = \fint_{B_\varepsilon^{\pi_{\nu_{\min}}}} u(x+h)\,d\mathcal{L}^{n-1}(h) + o\big(\varepsilon^2\big),$$

where $u(x + \varepsilon\nu_{\min}) = \min_{y\in\bar B_\varepsilon(x)} u(y)$, and $B_\varepsilon^{\pi_{\nu_{\min}}}$ is the $(n-1)$-dimensional ball centered at zero in the hyperplane $\pi_{\nu_{\min}}$, which is perpendicular to $\nu_{\min}$. Both definitions are to be understood in the viscosity sense. In Theorems 11 and 13, we extend this formula to the whole p-range by interpolating between the 1-Laplacian and the infinity Laplacian. Finally, we state the dynamic programming principle in Lemma 14.

In Section 2 we review the various definitions of viscosity solutions for the p-Laplacian. In Section 3 we study the limit case $p = 1$. The results of this section are used in Section 4, where we consider p-harmonic functions in the sense of averages for all $p > 1$. The corresponding asymptotic mean value characterization is derived in Section Section 5. Finally in Section 6 we describe a tug of war game whose value function satisfies an appropriate dynamic programming principle.

2. Viscosity solutions of the normalized p-Laplacian

Let u be a real valued function of class C^2 with nonvanishing gradient. For $p \in [1, \infty)$ the normalized version of the p-Laplace operator acting on u is

$$\Delta_p^N u = \frac{1}{p}|\nabla u|^{2-p}\Delta_p u = \frac{1}{p}|\nabla u|^{2-p}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)$$

while for $p = \infty$ we set

$$\Delta_\infty^N u = |\nabla u|^{-2}\Delta_\infty u = |\nabla u|^{-2}\bigl\langle D^2 u\nabla u, \nabla u\bigr\rangle.$$

After a calculation we see that

$$\begin{aligned}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) &= |\nabla u|^{p-2}\bigl((p-2)|\nabla u|^{-2}\bigl\langle D^2 u\nabla u, \nabla u\bigr\rangle + \Delta u\bigr)\\ &= |\nabla u|^{p-2}\bigl((p-2)\Delta_\infty^N + \Delta u\bigr)\\ &= |\nabla u|^{p-2}\bigl((p-1)\Delta_\infty^N u + \bigl(\Delta u - \Delta_\infty^N u\bigr)\bigr)\\ &= |\nabla u|^{p-2}\bigl((p-1)\Delta_\infty^N u + \Delta_1^N u\bigr),\end{aligned} \tag{6}$$

and thus

$$\Delta_p^N u = \frac{1}{p}\Delta_1^N u + \frac{1}{q}\Delta_\infty^N, \tag{7}$$

where q is the Hölder conjugate of p, $1/p + 1/q = 1$. Note that $\Delta_2^N = (1/2)\Delta u$ and that

$$\Delta u = \Delta_1^N + \Delta_\infty^N.$$

As the name viscosity solution suggests, one of their origins lies in adding an artificial viscosity term $\varepsilon\Delta u$ to a degenerate elliptic equation and sending ε to zero. For the normalized 1-Laplacian this amounts to studying equations like $\varepsilon\Delta u_\varepsilon + \Delta_1^N u_\varepsilon = 0$ and this equation can be rewritten as $(2\varepsilon + 1)\Delta_{p_\varepsilon}^N u_\varepsilon = 0$ with $p_\varepsilon = 1 + \frac{\varepsilon}{1+\varepsilon}$. Notice that $p_\varepsilon \to 1$ as $\varepsilon \to 0$. In fact, it is known that a sequence of p-harmonic functions converges to a 1-harmonic function as $p \to 1$; see [6] and also the discussion before Theorem 6. Moreover, a special 1-harmonic limit is chosen. It is a function of least gradient as pointed out by Juutinen in [10, Remark 3.3].

To deal with the case of nonsmooth functions, we define $\mathbb{F}_p(\eta, X)$ for $\eta \in \mathbb{R}^n \setminus \{0\}$ and for symmetric $n \times n$ matrices X as

$$\mathbb{F}_p(\eta, X) = \sum_{i,j}^{n}\left(\frac{1}{p}\delta_{ij} + \frac{(p-2)}{p}\frac{\eta_i\eta_j}{|\eta|^2}\right)X_{ij},$$

and for $p = \infty$ as

$$\mathbb{F}_\infty(\eta, X) = \sum_{i,j}^{n}\left(\frac{\eta_i\eta_j}{|\eta|^2}\right)X_{ij},$$

so that we always have

$$\mathbb{F}_p(\eta, X) = \frac{1}{p}\mathbb{F}_1(\eta, X) + \frac{1}{q}\mathbb{F}_\infty(\eta, X).$$

These functions are used to define viscosity solutions to the nonhomogeneous problem for the p-Laplace operator in nondivergence form

$$\begin{aligned}\Delta_p^N u(x) &= \left[\frac{1}{p}|\nabla u|^{2-p}\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)\right](x)\\ &= \mathbb{F}_p\bigl(\nabla u(x), D^2 u(x)\bigr) = g(x).\end{aligned} \tag{8}$$

To define a viscosity solution to the equation

$$\mathbb{F}_p\big(\nabla u, D^2 u\big) = g(x) \tag{9}$$

with $g \in C(\bar{\Omega})$, $g > 0$ (or $g < 0$), we need to compute $\mathbb{F}_p(\nabla\phi(x), D^2\phi(x))$ for C^2-smooth test functions touching u from above or below at the test point $x \in \Omega$. Unfortunately, except for $p = 2$, the functions $\mathbb{F}_p(\eta, X)$ are discontinuous when $\eta = 0$. There are several ways in the literature, see for example [6,9,23], to resolve this difficulty.

1. We can modify our requirements when $\nabla\phi(x) = 0$,
2. we can restrict the class of test functions so that $\mathbb{F}_p(\nabla\phi(x), D^2\phi(x))$ is uniquely defined also when $\nabla\phi(x) = 0$ by $\lim_{\eta\to 0}\mathbb{F}_p(\eta, D^2\phi(x))$, and
3. we can extend the domain of $\mathbb{F}_p(\eta, X)$ by using semicontinuous extensions.

Let us start with the first approach.

Definition 1. A continuous function v is a viscosity solution to the equation

$$\mathbb{F}_p\big(\nabla v, D^2 v\big) = g(x)$$

at x, if and only if every C^2-function ϕ that touches v from below in x satisfies

$$\mathbb{F}_p\big(\eta, D^2\phi(x)\big) \leqslant g(x) \quad \begin{cases} \text{for } \eta = \nabla\phi(x) & \text{if } \nabla\phi(x) \neq 0, \\ \text{for some } \eta \subset B_1(0)\setminus\{0\} & \text{if } \nabla\phi(x) = 0, \end{cases}$$

and every C^2-function ϕ that touches u from above at x satisfies

$$\mathbb{F}_p\big(\eta, D^2\phi(x)\big) \geqslant g(x) \quad \begin{cases} \text{for } \eta = \nabla\phi(x) & \text{if } \nabla\phi(x) \neq 0, \\ \text{for some } \eta \subset B_1(0)\setminus\{0\} & \text{if } \nabla\phi(x) = 0. \end{cases}$$

By saying that ϕ touches u from below at x_0, we mean

i) $u(x_0) = \phi(x_0)$,
ii) $u(x) > \phi(x)$ for $x \in \Omega,\ x \neq x_0$.

Alternatively, we could require that $u - \phi$ has a strict local minimum at x_0. If no such test function exists, nothing is required. The lower-semicontinuous functions satisfying the first half of the definition are called supersolutions, and the upper-semicontinuous functions satisfying the second half are called subsolutions.

Given a point $x \in \Omega$ we consider the class of good test functions

$$A(x) = \big\{\phi \in C^2 \text{ with } \nabla\phi(x) \neq 0 \text{ or } D^2\phi(x) = \mathbf{0}\big\}.$$

When $\phi \in A(x)$ we can always uniquely define $\mathbb{F}(\nabla\phi(x), D^2\phi(x))$. When $\nabla\phi(x) = 0$ we set

$$\Delta_p^N\phi(x) = \lim_{\eta\to 0}\mathbb{F}_p(\eta, \mathbf{0}) = 0. \tag{10}$$

Definition 2. A continuous function v is a viscosity solution to the equation

$$\mathbb{F}_p\big(\nabla v, D^2 v\big) = g(x)$$

at x, if and only if every C^2-function $\phi \in A(x)$ that touches v from below in x satisfies

$$\mathbb{F}_p\big(\nabla\phi(x), D^2\phi(x)\big) \leqslant g(x)$$

and every C^2-function $\phi \in A(x)$ that touches u from above at x satisfies

$$\mathbb{F}_p\big(\nabla\phi(x), D^2\phi(x)\big) \geqslant g(x).$$

To state our third definition we need the semicontinuous extensions of $\mathbb{F}_p$. For a symmetric matrix X we denote by $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ the smallest and largest eigenvalues of X respectively. For $p > 2$ the upper-semicontinuous extension is given by

$$\mathbb{F}_p^*(0, X) = \limsup_{\eta \to 0} \mathbb{F}_p(\eta, X) = \frac{1}{p}\,\mathrm{trace}(X) + \frac{(p-2)}{p}\lambda_{\max}(X)$$

and the lower-semicontinuous extension by

$$\mathbb{F}_{*,p}(0, X) = \liminf_{\eta \to 0} \mathbb{F}_p(\eta, X) = \frac{1}{p}\,\mathrm{trace}(X) + \frac{(p-2)}{p}\lambda_{\min}(X).$$

Otherwise, we define $\mathbb{F}_p^* = \mathbb{F}_{*,p} = \mathbb{F}_p$. For $p < 2$ we need to exchange $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$. Observe that

$$-\infty < \mathbb{F}_{*,p}(0, X) \leqslant \mathbb{F}_p^*(0, X) < \infty,$$

and that for $\phi \in A(x)$ we have

$$\mathbb{F}_{*,p}\big(\nabla\phi(x), D^2\phi(x)\big) = \mathbb{F}_p^*\big(\nabla\phi(x), D^2\phi(x)\big).$$

Definition 3. A continuous function v is a viscosity solution to the equation

$$\mathbb{F}_p\big(\nabla v, D^2 v\big) = g(x)$$

at x, if and only if every C^2-function ϕ that touches v from below in x satisfies

$$\mathbb{F}_{*,p}\big(\nabla\phi(x), D^2\phi(x)\big) \leqslant g(x)$$

and every C^2-function ϕ that touches u from above at x satisfies

$$\mathbb{F}_p^*\big(\nabla\phi(x), D^2\phi(x)\big) \geqslant g(x).$$

The above definitions are equivalent. The proof of this fact is based on the well-known fourth order perturbation argument, cf. [4,8] or [11].

Proposition 4. *Definitions* 1, 2, *and* 3 *are equivalent for* $1 \leqslant p \leqslant \infty$ *and* $g \in C(\bar{\Omega})$, $g > 0$ (*or* $g < 0$).

Proof. We restrict ourselves to the case of finite p, since the case $p = \infty$ follows by a simple modification. Clearly Definitions 1 and 3 are equivalent, and we can focus attention on showing that we can restrict the class of test functions as in Definition 2. We show directly that if Definition 3 fails, then also Definition 2 fails. To this end, we suppose that there is $\phi \in C^2(\bar{\Omega})$ and $x_0 \in \Omega$ such that

i) $u(x_0) = \phi(x_0)$,
ii) $u(x) > \phi(x)$ for $x \in \Omega,\ x \neq x_0$,

for which $\nabla\phi(x_0) = 0$, and

$$pg(x_0) < \lambda_{\min}\big((p-2)D^2\phi(x_0)\big) + \Delta\phi(x_0). \tag{11}$$

We then go on showing that there exists a test function ϕ with either $\nabla\phi(x) \neq 0$ or $\nabla\phi(x) = 0$, $D^2\phi(x) = \mathbf{0}$, for which the definition of a viscosity solution fails.

Let $\delta > 0$ be small, and set

$$w_j(x, y) = (1-\delta)u(x) - \left(\phi(y) - \frac{j}{4}|x-y|^4\right),$$

and denote by (x_j, y_j) the minimum point of w_j in $\bar{\Omega} \times \bar{\Omega}$. Since x_0 is a strict local minimum for $u - \phi$ there exists a strict local minimum x_0^δ for $(1-\delta)u - \phi$ and small enough $\delta > 0$ such that $x_0^\delta \to x_0$ as $\delta \to 0$. By first choosing a small enough $\delta > 0$ and then large enough j, we have $x_j, y_j \in \Omega$, and

$$x_j, y_j \to x_0^\delta, \quad \text{as } j \to \infty.$$

We observe that

$$\phi(y) - \frac{j}{4}|x_j - y|^4,$$

has a local maximum at y_j. By (11) and continuity of

$$x \mapsto \lambda_{\min}\big((p-2)D^2\phi(x)\big) + \Delta\phi(x),$$

and g, we have

$$pg(y_j) < \lambda_{\min}\big((p-2)D^2\phi(y_j)\big) + \Delta\phi(y_j) \tag{12}$$

for small enough $\delta > 0$ and large enough j. We denote $\varphi = \frac{j}{4}|x_j - y|^4$, and observe that $D^2\phi(y_j) \leqslant D^2\varphi(y_j)$. Thus by (12) we have

$$pg(y_j) < \lambda_{\min}\big((p-2)D^2\varphi(y_j)\big) + \Delta\varphi(y_j). \tag{13}$$

This also holds when $p < 2$, because

$$\begin{aligned}
\lambda_{\min}\big((p-2)D^2\phi(y_j)\big) + \Delta\phi(y_j) &= (p-2)\lambda_{\max}\big(D^2\phi(y_j)\big) + \operatorname{trace}\big(D^2\phi(y_j)\big) \\
&= (p-1)\lambda_{\max}\big(D^2\phi(y_j)\big) + \sum_{\lambda_i \neq \lambda_{\max}} \lambda_i\big(D^2\phi(y_j)\big) \\
&\leqslant (p-1)\lambda_{\max}\big(D^2\varphi(y_j)\big) + \sum_{\lambda_i \neq \lambda_{\max}} \lambda_i\big(D^2\varphi(y_j)\big).
\end{aligned}$$

We consider the two cases: either $x_j \neq y_j$ for all j large enough or $x_j = y_j$ infinitely often. First, let $y_j \neq x_j$. We use the theorem of sums for w_j, see [5] and also [6]. It implies that there exists symmetric matrices X_j, Y_j such that $X_j - Y_j$ is positive semidefinite, and

$$\begin{aligned}
&\big(j|x_j - y_j|^2(x_j - y_j), Y_j\big) \in \bar{J}^{2,+}\phi(y_j), \\
&\big(j|x_j - y_j|^2(x_j - y_j), X_j\big) \in \bar{J}^{2,-}u_\delta(x_j),
\end{aligned}$$

where we denoted $u_\delta = (1-\delta)u$. Using this fact, inequality (12), the continuity of g, and the fact that $g > 0$ in Ω, we get for large enough j that

$$\begin{aligned}
(1-\delta)pg(x_j) &< pg(y_j) \\
&< (p-2)\left\langle Y_j\frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\right\rangle + \operatorname{trace}(Y_j) \\
&\leqslant (p-2)\left\langle X_j\frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\right\rangle + \operatorname{trace}(X_j)
\end{aligned}$$

so that the definition of the viscosity solution fails already for $(j|x_j - y_j|^2(x_j - y_j), X_j) \in \bar{J}^{2,-}u_\delta(x_j)$ with non-vanishing $j|x_j - y_j|^2(x_j - y_j)$. In the case $g < 0$ in Ω we need to replace $1-\delta$ by $1+\delta$ throughout the argument. If $p < 2$, the last inequality follows from the calculation

$$\begin{aligned}
&(p-2)\left\langle (Y_j - X_j)\frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\right\rangle + \operatorname{trace}(Y_j - X_j) \\
&\quad\leqslant (p-2)\lambda_{\min} + \sum_{i=1}^{n}\lambda_i \\
&\quad= (p-1)\lambda_{\min} + \sum_{\lambda_i \neq \lambda_{\min}} \lambda_i \\
&\quad\leqslant 0,
\end{aligned}$$

where λ_i, $\lambda_{\min}$, and $\lambda_{\max}$ denote the eigenvalues of $Y_j - X_j$.

Let then $x_j = y_j$. The fact $D^2\varphi(y_j) = D^2(\frac{j}{4}|x_j - y_j|^4) = 0$ together with (13) shows that this case cannot happen. If g were negative instead of positive, this case would show that there exists a test function with $\nabla\varphi(y_j) = 0$, $D^2\varphi(y_j) = \mathbf{0}$ for which Definition 2 fails. □

A similar argument also provides comparison principle and uniqueness, see also Lu and Wang [17,18]. Notice that Theorem 5 below is only stated for $g > 0$. In fact, for $p = 1$ and $g \equiv 0$, there is a counterexample in [25], see also [13] and [14].

Theorem 5. *Let Ω be a bounded domain, u lower-semicontinuous and v upper-semicontinuous. Suppose that v is a subsolution, and u a supersolution to* (9) *with $g \in C(\bar{\Omega})$, $g > 0$ and $1 \leqslant p \leqslant \infty$. Further, suppose that $v \leqslant u$ on $\partial\Omega$ in the sense that*

$$\limsup_{x\to z} v(x) \leqslant \liminf_{x\to z} u(z) \tag{14}$$

for all $z \in \partial\Omega$, where both sides are not simultaneously $-\infty$ or ∞. Then

$$v \leqslant u \quad \text{in } \Omega.$$

Proof. We consider first the case $2 \leqslant p < \infty$. We argue by contradiction and assume that $u - v$ has a strict interior minimum, that is,

$$u(x_0) - v(x_0) = \inf_{\Omega}(u - v) < \inf_{\partial\Omega}(u - v).$$

Let $\delta \in (0, 1)$, and set

$$w_j(x, y) = (1 - \delta)u(x) - \left(v(y) - \frac{j}{4}|x - y|^4\right),$$

and denote by (x_j, y_j) the minimum point of w_j in $\bar{\Omega} \times \bar{\Omega}$. Since x_0 is a local minimum for $u - v$, there exists a strict local minimum x_0^δ for $(1 - \delta)u - v$ such that $x_0^\delta \to x_0$ as $\delta \to 0$. Further

$$x_j, y_j \to x_0^\delta, \quad \text{as } j \to \infty,$$

and $x_j, y_j \in \Omega$ for all large j. It follows that

$$y \mapsto v(y) - \frac{j}{4}|x_j - y|^4,$$

has a local maximum at y_j, and

$$x \mapsto (1 - \delta)u(x) + \frac{j}{4}|x - y_j|^4,$$

a local minimum at x_j.

Observe that if $y_j = x_j$, then $\nabla\varphi(y_j) = 0$, $D^2\varphi(y_j) = \mathbf{0}$, which immediately contradicts with the subsolution property of v since $g > 0$. Thus we may concentrate on the case $x_j \neq y_j$. Again, theorem of sums for w_j implies that there exists symmetric matrices X_j, Y_j such that $X_j - Y_j$ is positive semidefinite, and

$$\begin{aligned} &\left(j|x_j - y_j|^2(x_j - y_j), Y_j\right) \in \bar{J}^{2,+}v(y_j), \\ &\left(j|x_j - y_j|^2(x_j - y_j), X_j\right) \in \bar{J}^{2,-}u_\delta(x_j), \end{aligned}$$

where we denoted $u_\delta = (1 - \delta)u$ so that u_δ satisfies $(1 - \delta)g \geqslant \Delta_p^N u_\delta$ in the viscosity sense, i.e.

$$(1 - \delta)pg(x_j) \geqslant (p - 2)\left\langle X_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|}\right\rangle + \operatorname{trace}(X_j).$$

Using this and the corresponding inequality for v, we get for large enough j that

$$
\begin{aligned}
0 &< pg(y_j) - (1-\delta)pg(x_j) \\
&\leqslant (p-2)\left\langle Y_j \frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\right\rangle + \operatorname{trace}(Y_j) \\
&\quad - (p-2)\left\langle X_j \frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\right\rangle - \operatorname{trace}(X_j) \\
&= (p-2)\left\langle (Y_j - X_j) \frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\right\rangle + \operatorname{trace}(Y_j - X_j) \\
&\leqslant 0,
\end{aligned}
$$

because $Y_j - X_j$ is negative semidefinite. In the first inequality we used continuity of g. This provides the desired contradiction. The cases $1 \leqslant p < 2$ and $p = \infty$ can be treated in a similar fashion as above. □

According to [12], when $1 < p < \infty$ and $g \equiv 0$, it is enough to test using test functions with $\nabla\phi(x) \neq 0$. This definition still guarantees the uniqueness. We observe that the proof of Proposition 4 shows that in the case $g \equiv 0$ and $1 \leqslant p \leqslant \infty$, Definition 3 is equivalent to a definition where we only use test functions with $\nabla\phi(x) \neq 0$.

Theorem 6. *A continuous function v is a viscosity solution to the equation*

$$\mathbb{F}_p\big(\nabla v, D^2 v\big) = 0, \quad 1 \leqslant p \leqslant \infty$$

at x, if and only if every C^2-function ϕ with $\nabla\phi(x) \neq 0$ that touches v from below in x satisfies

$$\mathbb{F}_p\big(\nabla\phi(x), D^2\phi(x)\big) \leqslant 0$$

and every C^2-function ϕ, $\nabla\phi(x) \neq 0$ that touches u from above at x satisfies

$$\mathbb{F}_p\big(\nabla\phi(x), D^2\phi(x)\big) \geqslant 0.$$

3. 1-harmonic functions in the sense of averages

Given a unit vector $\nu \in \mathbb{R}^n$ consider the $(n-1)$-dimensional hyperplane

$$\pi = \nu^\perp = \big\{x \in \mathbb{R}^n\colon \langle x, \nu\rangle = 0\big\}.$$

For small $\varepsilon > 0$ we denote by B_ε^π the $(n-1)$-dimensional ball in π centered at 0 with radius ε

$$B_\varepsilon^\pi = B_\varepsilon(0) \cap \pi.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ be a C^2-function.

Averaging the Taylor expansion

$$u(x+h) = u(x) + \big\langle \nabla u(x), h\big\rangle + \frac{1}{2}\big\langle D^2u(x)h, h\big\rangle + o\big(|h|^2\big), \tag{15}$$

over B_ε^π we obtain

$$\fint_{B_\varepsilon^\pi} u(x+h)\, d\mathcal{L}^{n-1}(h) = u(x) + \varepsilon^2 \cdot \frac{1}{2(n+1)} \Delta_\pi u(x) + o\big(\varepsilon^2\big), \tag{16}$$

where Δ_π denotes the Laplace operator on the plane $x + \pi$. To see this, we use the orthonormal basis made up of ν and an orthonormal basis for π, and observe that

$$\frac{1}{2}\fint_{B_\varepsilon^\pi} \big\langle D^2u(x)h, h\big\rangle\, d\mathcal{L}^{n-1}(h) = \varepsilon^2 \cdot \frac{1}{2(n+1)} \Delta_\pi u(x),$$

cf. [19]. We denote by $D_{\nu\nu}^2 u(x) = \langle D^2u(x)\nu, \nu\rangle$ the second derivative of u at x in the direction ν. Note that

$$\begin{aligned}\Delta u(x) &= \operatorname{trace}\big(D^2u(x)\big) = \Delta_\pi u(x) + D^2_{\nu\nu}u(x)\\ &= \Delta_\pi u(x) + \big\langle D^2u(x)\nu, \nu\big\rangle.\end{aligned}$$

Thus we get a formula for Δ_π

$$\Delta_\pi u(x) = \Delta u(x) - \big\langle D^2u(x)\nu, \nu\big\rangle. \tag{17}$$

Suppose that $\nabla u(x) \neq 0$, and write

$$\nu = -\frac{\nabla u(x)}{|\nabla u(x)|}.$$

The vector ν is the exterior normal to the level set

$$S = \big\{y \in \mathbb{R}^n\colon\ u(y) \geqslant u(x)\big\}.$$

Whenever $\nabla u(x)$ is nonzero, the mean curvature $H(x)$ of S is given by

$$H(x) = \frac{1}{n-1}\operatorname{div}(-\nu),$$

so that we can rewrite $\Delta_\pi u(x)$ as

$$\begin{aligned}\Delta_\pi u(x) = \Delta_1^N u(x) &= \big|\nabla u(x)\big|\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)(x)\\ &= (n-1)H(x)\big|\nabla u(x)\big|.\end{aligned} \tag{18}$$

Here Δ_1^N refers to the notation introduced in (8).

Eq. (16) immediately implies a characterization of harmonic functions on the hyperplane in a sense of averages.

Proposition 7. *Let $u \in C^2(\Omega)$, $\nu \in \mathbb{R}^n$ be a unit vector, and π the $(n-1)$-dimensional hyperplane defined by ν. Then*

$$\fint_{B_\varepsilon^\pi} u(x+h)\,d\mathcal{L}^{n-1}(h) = u(x) + o\big(\varepsilon^2\big)$$

if and only if $\Delta_\pi u(x) = 0$.

We define unit vectors $\nu_{\min}$ and $\nu_{\max}$ by requiring that

$$\begin{aligned}u(x+\varepsilon\nu_{\min}) &= \min_{y\in\bar{B}_\varepsilon(x)} u(y),\\ u(x+\varepsilon\nu_{\max}) &= \max_{y\in\bar{B}_\varepsilon(x)} u(y).\end{aligned} \tag{19}$$

Observe that whenever $\nabla u(x) \neq 0$, then $\nu_{\min}$ and $\nu_{\max}$ converge to the uniquely defined directions

$$-\frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{and} \quad \frac{\nabla u(x)}{|\nabla u(x)|}$$

respectively even if those vectors themselves may not be unique.

Definition 8. A continuous function u is 1-harmonic in the sense of averages if

$$u(x) = \fint_{B_\varepsilon^{\pi_\nu}} u(x+h)\,d\mathcal{L}^{n-1}(h) + o\big(\varepsilon^2\big) \quad \text{as } \varepsilon \to 0, \tag{20}$$

in the viscosity sense, i.e.

1. if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ from below, we have
$$\phi(x) \geqslant \fint_{B_\varepsilon^{\pi\nu_{\max}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) + o(\varepsilon^2),$$
for any $\nu_{\max}$ in (19) as $\varepsilon \to 0$,
2. if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ from above, we have
$$\phi(x) \leqslant \fint_{B_\varepsilon^{\pi\nu_{\min}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) + o(\varepsilon^2),$$
for any $\nu_{\min}$ in (19) as $\varepsilon \to 0$.

Theorem 9. *A continuous function u in a domain $\Omega \subset \mathbb{R}^n$ is 1-harmonic in the sense of averages if and only if*
$$\Delta_1^N u = 0$$
in the viscosity sense.

Proof. For a smooth ϕ
$$\fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) = \phi(x) + \frac{\varepsilon^2}{2(n+1)} \Delta_{\pi_\nu}\phi(x) + o(\varepsilon^2), \tag{21}$$
holds for any $\nu \neq 0$. If $\nabla\phi \neq 0$, then
$$\nu_{\min} \to -\nabla\phi/|\nabla\phi| = \nu \quad \text{as } \varepsilon \to 0. \tag{22}$$
We choose $\phi \in C^2,\ \nabla\phi \neq 0$ such that ϕ touches u at $x \in \Omega$ from below, and suppose that u is a viscosity solution to $\Delta u - \Delta_\infty^N u = \Delta_{\pi_\nu} u = 0$. In particular, $\Delta_{\pi_\nu}\phi(x) \leqslant 0$. This, (21), (22), and the continuity of the second derivatives imply
$$\fint_{B_\varepsilon^{\pi\nu_{\min}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \leqslant \phi(x) + o(\varepsilon^2).$$
The second half of the definition of a viscosity solution follows similarly.

Suppose then that u is 1-harmonic in the sense of averages, i.e. for the above ϕ we have
$$\phi(x) \geqslant \fint_{B_\varepsilon^{\pi\nu_{\min}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) + o(\varepsilon^2).$$
Combining this together with (21) we obtain
$$0 \geqslant \frac{\varepsilon^2}{2(n+1)} \Delta_{\pi_{\nu_{\min}}}\phi(x) + o(\varepsilon^2).$$
Dividing this by ε^2, passing to a limit with ε, and using (22), we see that u satisfies the condition for the viscosity supersolution with this ϕ. The proof for the second half is analogous.

We are left with the case $\nabla\phi(x) = 0$. Suppose that $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ from below with $\nabla\phi(x) = 0$, and suppose that u is a viscosity solution so that
$$-\lambda_{\max}(D^2\phi(x)) + \Delta\phi(x) \leqslant 0.$$
Observing that now
$$\langle D^2\phi\nu_{\max}, \nu_{\max}\rangle \to \lambda_{\max}(D^2\phi(x)) \tag{23}$$
as $\varepsilon \to 0$, and combining this with (21), we see that u is 1-harmonic in the sense of averages.

Suppose then that u is 1-harmonic in the sense of averages, i.e. for the above ϕ we have

$$\phi(x) \geqslant \fint_{B_\varepsilon^{\pi\nu_{\max}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) + o\big(\varepsilon^2\big).$$

Combining this together with (21) and (23), dividing by ε^2 and passing to a limit with ε, we see that u satisfies the first half of a definition of a viscosity solution. The second half is again analogous. $\square$

4. p-harmonic functions in the sense of averages

We start with a formal calculation assuming that u is smooth and $\nabla u \neq 0$. The gradient direction is almost the maximizing direction for a smooth function whenever the gradient does not vanish. We insert $h = \pm\nabla u/|\nabla u|$ in (15) and sum up the two resulting expansions to get rid of the first order terms

$$\begin{aligned} u(x) - \frac{1}{2}\Big\{\max_{\bar{B}_\varepsilon(x)} u + \min_{\bar{B}_\varepsilon(x)} u\Big\} &\approx u(x) - \frac{1}{2}\Big\{u\Big(x + \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\Big) + u\Big(x - \varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\Big)\Big\} \\ &= -\frac{\varepsilon^2}{2}\Delta_\infty^N u(x) + o\big(\varepsilon^2\big). \end{aligned} \tag{24}$$

Next we multiply (24) and (16) by the constants $\alpha = \frac{p-1}{p+n}$ and $\beta = \frac{n+1}{p+n}$ satisfying $\alpha + \beta = 1$ and add up the formulas so that we have the operator in (6) on the right hand side. We get

$$\begin{aligned} &\frac{n+1}{p+n} \fint_{B_\varepsilon^{\pi\nu_{\min}}} u(x+h)\, d\mathcal{L}^{n-1}(h) + \Big(\frac{p-1}{p+n}\Big)\frac{\max_{\bar{B}_\varepsilon(x)} u + \min_{\bar{B}_\varepsilon(x)} u}{2} \\ &\quad\approx u(x) + \frac{\varepsilon^2}{2(p+n)}\big((p-1)\Delta_\infty^N u + \Delta_1^N u(x)\big) + o\big(\varepsilon^2\big). \end{aligned}$$

This motivates the following definition which we only formulate in the case $1 \leqslant p \leqslant 2$. In the case $p > 2$ the definition is almost identical, except that $\nu_{\min}$ and $\nu_{\max}$ should be interchanged. This only plays a role when $\nabla\phi$ vanishes. In that case

$$\big\langle D^2\phi\nu_{\max}, \nu_{\max}\big\rangle \to \lambda_{\max}\big(D^2\phi(x)\big) \quad \text{and} \quad \big\langle D^2\phi\nu_{\min}, \nu_{\min}\big\rangle \to \lambda_{\min}\big(D^2\phi(x)\big)$$

see also (23).

Definition 10. A continuous function u is p-harmonic, $1 \leqslant p \leqslant 2$, in the sense of averages if it satisfies

$$\begin{aligned} u(x) = {} & \frac{n+1}{p+n} \fint_{B_\varepsilon^{\pi\nu}} u(x+h)\, d\mathcal{L}^{n-1}(h) + \Big(\frac{p-1}{p+n}\Big)\frac{\max_{\bar{B}_\varepsilon(x)} u + \min_{\bar{B}_\varepsilon(x)} u}{2} \\ & + o\big(\varepsilon^2\big), \end{aligned} \tag{25}$$

as $\varepsilon \to 0$ *in the viscosity sense*, that is,

1. for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ from below, we have

   $$\begin{aligned} \phi(x) \geqslant {} & \frac{n+1}{p+n} \fint_{B_\varepsilon^{\pi\nu_{\max}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) + \Big(\frac{p-1}{p+n}\Big)\frac{\max_{\bar{B}_\varepsilon(x)} \phi + \min_{\bar{B}_\varepsilon(x)} \phi}{2} \\ & + o\big(\varepsilon^2\big), \end{aligned}$$

   for any $\nu_{\max}$ in (19) as $\varepsilon \to 0$,

2. for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ from above, we have

$$\phi(x) \leqslant \frac{n+1}{p+n} \fint_{B_\varepsilon^{\pi\nu_{\min}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) + \left(\frac{p-1}{p+n}\right) \frac{\max_{\bar{B}_\varepsilon(x)} \phi + \min_{\bar{B}_\varepsilon(x)} \phi}{2} + o\left(\varepsilon^2\right),$$

for any $\nu_{\min}$ in (19) as $\varepsilon \to 0$.

Unfortunately, using test functions in the above definition instead of u itself seems to be necessary to obtain the next theorem as indicated by the known counterexample in the case $p = \infty$ (see [19] for an example).

For $p = 1$, Theorem 11 follows from Theorem 9, for $p = \infty$, the proof follows from (5), cf. Theorem 2 in [19], and for $p \in (1, \infty)$ it is analogous to the proof of Theorem 13 below, so we omit it here.

Theorem 11. *Let $1 \leqslant p \leqslant \infty$. A continuous function u in a domain $\Omega \subset \mathbb{R}^n$ is p-harmonic in the sense of averages according to Definition* 10 *if and only if*

$$\Delta_p^N u(x) = 0$$

in the viscosity sense.

5. Mean value formula for a tug-of-war game

We already defined p-harmonic functions in the sense of averages in the previous section. In this section, we derive another mean value formula. It appears to be more complicated, but in the context of the tug-of-war game similar to that in [22] it turns out to be quite natural. Below

$$\alpha = \frac{p-1}{p+n}, \qquad \beta = \frac{n+1}{p+n},$$

and $B_\varepsilon^{\pi_\nu}$ is the $(n-1)$-dimensional ball of radius ε centered at zero in the hyperplane π_ν.

Definition 12. A continuous function u satisfies

$$u(x) = \frac{1}{2} \sup_{0<|\nu|\leqslant 1} \left\{ \alpha u(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi_\nu}} u(x+h)\, d\mathcal{L}^{n-1}(h) \right\} + \frac{1}{2} \inf_{0<|\nu|\leqslant 1} \left\{ \alpha u(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi_\nu}} u(x+h)\, d\mathcal{L}^{n-1}(h) \right\} + o\left(\varepsilon^2\right), \tag{26}$$

as $\varepsilon \to 0$ *in the sense of averages* if

1. for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ from below with $\nabla\phi(x) \neq 0$, we have

$$\phi(x) \geqslant \frac{1}{2} \sup_{0<|\nu|\leqslant 1} \left\{ \alpha \phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi_\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\} + \frac{1}{2} \inf_{0<|\nu|\leqslant 1} \left\{ \alpha \phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi_\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\} + o\left(\varepsilon^2\right),$$

2. for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ from above with $\nabla\phi(x) \neq 0$, we have

$$\phi(x) \leqslant \frac{1}{2} \sup_{0<|\nu|\leqslant 1} \left\{ \alpha\phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\} + \frac{1}{2} \inf_{0<|\nu|\leqslant 1} \left\{ \alpha\phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\} + o\big(\varepsilon^2\big).$$

The case $p=\infty$ is already considered in [19], and thus we concentrate on the case $1<p<\infty$.

Theorem 13. *Let $1<p<\infty$. A continuous function u in a domain $\Omega \subset \mathbb{R}^n$ satisfies Definition* 12 *if and only if*

$$\Delta_p^N u(x) = 0$$

in the viscosity sense.

Proof. First we recall a calculation from [19] leading to an asymptotic expansion involving the infinity Laplacian. Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Let $\nu_{\min}$ be a vector giving

$$\inf_{0<|\nu|\leqslant 1} \left\{ \alpha\phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\}.$$

Later we check that the infimum is not obtained when $\nu \to 0$ so that $\nu_{\min}$ really exists. Similarly, let $\nu_{\max}$ be a vector giving

$$\sup_{0<|\nu|\leqslant 1} \left\{ \alpha\phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\}.$$

Consider the Taylor expansion of the second order of ϕ

$$\phi(x+\nu) = \phi(x) + \nabla\phi(x)\cdot\nu + \frac{1}{2}\big\langle D^2\phi(x)\nu, \nu\big\rangle + o\big(|\nu|^2\big)$$

as $|\nu| \to 0$. Evaluating this Taylor expansion of ϕ at the point x with $\nu = \nu_{\min}\varepsilon$, and $\nu = -\nu_{\min}\varepsilon$ we get

$$\phi(x+\nu_{\min}\varepsilon) = \phi(x) + \nabla\phi(x)\cdot\nu_{\min}\varepsilon + \frac{1}{2}\big\langle D^2\phi(x)\nu_{\min}\varepsilon, \nu_{\min}\varepsilon\big\rangle + o\big(\varepsilon^2\big), \tag{27}$$

$$\phi(x-\nu_{\min}\varepsilon) = \phi(x) - \nabla\phi(x)\cdot\nu_{\min}\varepsilon + \frac{1}{2}\big\langle D^2\phi(x)\nu_{\min}\varepsilon, \nu_{\min}\varepsilon\big\rangle + o\big(\varepsilon^2\big) \tag{28}$$

as $\varepsilon \to 0$. Adding the expressions, we obtain

$$\phi(x-\nu_{\min}\varepsilon) + \phi(x+\nu_{\min}\varepsilon) - 2\phi(x) = \varepsilon^2\big\langle D^2\phi(x)\nu_{\min}, \nu_{\min}\big\rangle + o\big(\varepsilon^2\big). \tag{29}$$

We also have

$$\fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) = \phi(x) + \frac{\varepsilon^2}{2(n+1)}\Delta_{\pi_\nu}\phi(x) + o\big(\varepsilon^2\big) \tag{30}$$

for any nonzero ν. We choose $\nu = \nu_{\min}$ and $\nu = -\nu_{\min}$ in (30), multiply by $\alpha/2$ and sum up the results with (29) multiplied by $\beta/2$. Then we use the definition of $\nu_{\max}$ and estimate $\phi(x-\nu_{\min}\varepsilon) + \fint_{B_\varepsilon^{\pi-\nu_{\min}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h)$ from above, as well as observe that $\Delta_{\pi_{\nu_{\min}}} = \Delta_{\pi_{-\nu_{\min}}}$, and obtain

$$\frac{1}{2} \sup_{0<|\nu|\leqslant 1} \left\{ \alpha\phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\} + \frac{1}{2} \inf_{0<|\nu|\leqslant 1} \left\{ \alpha\phi(x+\nu\varepsilon) + \beta \fint_{B_\varepsilon^{\pi\nu}} \phi(x+h)\, d\mathcal{L}^{n-1}(h) \right\}$$

$$\geqslant \phi(x) + \frac{\beta\varepsilon^2}{2(n+1)}\big((p-1)\big\langle D^2\phi(x)\nu_{\min}, \nu_{\min}\big\rangle + \Delta_{\pi_{\nu_{\min}}}\phi(x)\big) + o\big(\varepsilon^2\big), \tag{31}$$

which holds for any smooth function for which $\nu_{\min} \neq 0$.

Suppose then that $\nabla\phi(x) \neq 0$. By considering the lowest order terms in the sum of (27) and (30)

$$\alpha\phi(x+\nu_{\min}\varepsilon) + \beta \fint_{B_\varepsilon^{\pi\nu_{\min}}} \phi(x+h)\, d\mathcal{L}^{n-1}(h)$$

$$= \phi(x) + \alpha\nabla\phi(x)\cdot\nu_{\min}\varepsilon + \alpha\frac{1}{2}\big\langle D^2\phi(x)\nu_{\min}\varepsilon, \nu_{\min}\varepsilon\big\rangle + \beta\frac{\varepsilon^2}{2(n+1)}\Delta_{\pi_{\nu_{\min}}}\phi(x) + o\big(\varepsilon^2\big),$$

we see that

$$\nu_{\min} \to -\nabla\phi/|\nabla\phi| \tag{32}$$

as $\varepsilon \to 0$ because $p \neq 1$, i.e. $\alpha > 0$.

Suppose that function u satisfies Definition 12. Consider a smooth ϕ, $\nabla\phi \neq 0$ which touches u at x from below. Combining Definition 12 together with (31), we obtain

$$0 \geqslant \frac{\beta\varepsilon^2}{2(n+1)}\big((p-1)\big\langle D^2\phi(x)\nu_{\min}, \nu_{\min}\big\rangle + \Delta_{\pi_{\nu_{\min}}}\phi(x)\big) + o\big(\varepsilon^2\big).$$

Dividing this by ε^2 and recalling (32), and passing to a limit, we obtain the condition in the definition for the viscosity supersolution because with $\nu = \nabla\phi/|\nabla\phi|$ we have $\Delta_1^N\phi(x) = \Delta_{\pi_\nu}\phi = \Delta\phi - \Delta_\infty^N\phi$.

To prove the reverse implication, assume that u is a viscosity solution. In particular u is a subsolution. Let ϕ be a smooth test function touching u at $x \in \Omega$ from above. If $\nabla\phi(x) \neq 0$ and we set $\nu = \nabla\phi/|\nabla\phi|$, it follows that

$$\begin{aligned} 0 \leqslant (p-2)\Delta_\infty^N\phi(x) + \Delta\phi(x) &= (p-1)\Delta_\infty^N\phi(x) + \big(\Delta\phi(x) - \Delta_\infty^N\phi(x)\big) \\ &= (p-1)\Delta_\infty^N\phi(x) + \Delta_{\pi_\nu}\phi(x). \end{aligned} \tag{33}$$

This together with (31) and (32) shows that ϕ satisfies the second half in Definition 12. The other cases are similar. According to Theorem 6, we only need to test with the test functions with nonvanishing gradients, and thus the proof is complete. □

6. Tug-of-war with noise

The asymptotic expansion in the previous section is related to the tug-of-war game with noise which is quite similar to that in [22].

Fix $\varepsilon > 0$ and consider the following two-player zero-sum-game. At the beginning, a token is placed at a point $x_0 \in \Omega$. First players fix their possible moves ν_{I} and ν_{II} with $|\nu_{\mathrm{I}}|, |\nu_{\mathrm{II}}| \leqslant \varepsilon$, and toss a fair coin. If Player I wins the toss, then she tosses a biased coin. If she gets heads (with probability $\alpha > 0$), the token is placed at $x_1 = x_0 + \nu_{\mathrm{I}}$. If she gets tails (with probability $\beta > 0$), then the token is placed at a random point in $x_1 \in B_\varepsilon^{\pi\nu_{\mathrm{I}}}(x_0)$. Similarly if Player II wins the toss, then he tosses a biased coin. If he gets heads (with probability α), the token is placed at $x_1 = x_0 + \nu_{\mathrm{II}}$. If he gets tails (with probability β), then the token is placed at a random point in $x_1 \in B_\varepsilon^{\pi\nu_{\mathrm{II}}}(x_0)$. The game is played until the token hits

$$\Gamma_\varepsilon = \big\{x \in \mathbb{R}^n \setminus \Omega \colon \operatorname{dist}(x, \partial\Omega) \leqslant \varepsilon\big\}.$$

This procedure yields a possibly infinite sequence of game states $x_0, x_1, \ldots$ where every x_k is a random variable. We denote by $x_\tau \in \Gamma_\varepsilon$ the first point in Γ_ε in the sequence, where τ is the hitting time. The payoff is $F(x_\tau)$, where $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given measurable *payoff function*. Player I earns $F(x_\tau)$ while Player II earns $-F(x_\tau)$.

A history of a game up to step k is a vector of the first $k+1$ game states and k steps, for example, $(x_0, \nu_1, x_1, \ldots, \nu_k, x_k)$. A strategy S_{I} for Player I is a Borel function defined on the space of all histories that gives the next step for Player I

$$\nu_{k+1}^{\mathrm{I}}, \quad \big|\nu_{k+1}^{\mathrm{I}}\big| \leqslant \varepsilon$$

given a history h if Player I wins the toss. Similarly Player II plays according to a strategy S_{II}.

Using the Kolmogorov construction the fixed starting point x_0 and the strategies S_{I} and S_{II} determine a unique probability measure $\mathbb{P}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}$.

The expected payoff, when starting from x_0 and using the strategies S_{I}, S_{II}, is

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\big[F(x_\tau)\big] = \int_{H^\infty} F\big(x_\tau(\omega)\big)\,d\mathbb{P}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}(\omega), \tag{34}$$

where we integrate over all histories H^∞.

The *value of the game for Player I* is given by

$$u^\varepsilon_{\mathrm{I}}(x_0) = \sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\big[F(x_\tau)\big]$$

while the *value of the game for Player II* is given by

$$u^\varepsilon_{\mathrm{II}}(x_0) = \inf_{S_{\mathrm{II}}}\sup_{S_{\mathrm{I}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\big[F(x_\tau)\big].$$

If the chosen strategies result in a game that does not end almost surely we set the expected pay-off for Player I to be $-\infty$ and for Player II to be $+\infty$. The values $u^\varepsilon_{\mathrm{I}}(x_0)$ and $u^\varepsilon_{\mathrm{II}}(x_0)$ are intuitively the best expected outcomes each player can guarantee when the game starts at x_0.

We start with the statement of the *Dynamic Programming Principle* (DPP) applied to our game.

Lemma 14 *(DPP). For $\alpha > 0$ and $\beta > 0$ (which corresponds to $1 < p < \infty$) the value function for Player I satisfies*

$$\begin{aligned} u^\varepsilon_{\mathrm{I}}(x) = {}& \frac{1}{2}\sup_{0<|\nu|\leqslant 1}\left\{\alpha u^\varepsilon_{\mathrm{I}}(x+\nu\varepsilon) + \beta \fint_{B^{\pi_\nu}_\varepsilon} u^\varepsilon_{\mathrm{I}}(x+h)\,d\mathcal{L}^{n-1}(h)\right\} \\ & + \frac{1}{2}\inf_{0<|\nu|\leqslant 1}\left\{\alpha u^\varepsilon_{\mathrm{I}}(x+\nu\varepsilon) + \beta \fint_{B^{\pi_\nu}_\varepsilon} u^\varepsilon_{\mathrm{I}}(x+h)\,d\mathcal{L}^{n-1}(h)\right\} \end{aligned} \tag{35}$$

for each $x_0 \in \Omega$ and

$$u^\varepsilon_{\mathrm{I}}(x_0) = F(x_0), \quad \text{for } x_0 \in \Gamma_\varepsilon.$$

The value function for Player II, $u^\varepsilon_{\mathrm{II}}$, satisfies the same equation.

An intuitive explanation for DPP can be obtained by considering the different outcomes of a single game round with the corresponding probabilities.

It turns out that the values of the game satisfy a *comparison principle*, the values are *unique*, $u_{\mathrm{I}} = u_{\mathrm{II}}$ with fixed boundary values and any function satisfying (35) is a game value. In a smooth domain with regular boundary data, the values converge to a unique p-harmonic function as the step size tends to zero. The proofs are similar to those in [21] and [22].

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